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Surveying in rotating systems

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Abstract. The principles of radar are applied to the problem of measuring distances in rotating systems. Observations made by a single observer rotating with an inertial angular velocity ω and at an inertial radius r are investigated and it is shown that the radius, as measured by the rotating observer, is given by $r' = r(1 - r^2\omega^2/c^2)^{1/2}$ where c is the velocity of light *in vacuo*. The angular velocity according to the rotating observer is shown to be $\omega' = \omega(1 - r^2\omega^2/c^2)^{-1/2}$. Also, piecemeal measurements of distances within rotating systems are made by summing an infinite number of infinitesimal, contiguous measurements that have been collated by an observer in the inertial frame of the centre of rotation of the system. Such measurements are used to determine the length of a light path between two points in the rotating system and to measure the shortest distance between two points in the rotating system. These two measurements are found not to be identical. It is also shown that light signals used to measure infinitesimal piecemeal distances in a rotating system are emitted and received, according to an observer rotating with the system, in one and the same direction.

1. Introduction

Recent experiments by Davies and Jennison (1974, 1975) of signals reflected by rotating mirrors have shown that the radius measured by a rotating observer must contract relative to that measured at the centre. This paper will be concerned with the analysis and some of the implications of this and similar phenomena in rotating systems.

It is currently acknowledged that radar measurements can give very precise values of the distance between two points and it may also be argued that such measurements, involving the relation, measured in proper time, between cause and effect, are the most physically meaningful definitions of distance. Using such radar measurements we investigate the consequences, within a rotating system, of the current standardization of the units of length and time (Sanders 1965) which imply the fundamental fact that: to every observer, whether or not accelerated, the speed of light *in vacuo* passing through his position in every direction is c. Measurements made by radar techniques are of necessity two-way measurements and distances are computed from the time interval that an electromagnetic signal takes for the two-way journey between two points. Therefore, in practice, $c\tau$ defines the unit of length, where τ is the proper local unit of time such as may be derived from the frequency of a local atomic standard clock.

As a rotating system is kinematic and not gravitational, it is not necessary to use general relativity in the following analysis.

2. Observations made by a single rotating observer at inertial radius r

Suppose we have a situation in which an observer is located at a constant distance r from the origin of an inertial system of coordinates, S, and that the observer is rotating with constant angular velocity ω with respect to S. If we take the cylindrical coordinates (r, θ, z, t) for S, with the plane $z = 0$ as the plane in which the observer rotates, then

$$ds^2 = dr^2 + r^2\, d\theta^2 + dz^2 - c^2\, dt^2$$

for the line element and, if τ is the proper time of the rotating observer and since $\omega = d\theta/dt$, we find that

$$ds^2 = -c^2\, d\tau^2 = r^2\omega^2\, dt^2 - c^2\, dt^2 = -c^2\, dt^2[1 - (r^2\omega^2/c^2)].$$

Hence,

$$\tau = t[1 - (r^2\omega^2/c^2)]^{1/2} \tag{1}$$

if ω and r are maintained constant during the measurement, and t is the inertial or central time (see for example Arzeliès 1966). Defining the 'radius' measured by the rotating observer as $r' = \frac{1}{2}c\tau_1$ gives us a method of relating the observer's radius to the inertial radius. τ_1 is the *time interval* of the standard atomic clock carried by the observer, between the emission of a light signal to the centre and its subsequent absorption by the same observer. The ray geometry in the inertial system is such that the signal is emitted at a point at radius r, scattered at the centre, and received at a different point at the same radius r. Thus in the inertial system $r = \frac{1}{2}ct_1$ where t_1 is the time interval between emission and reception as measured by a standard clock at rest relative to S. It therefore follows from equation (1) that

$$r' = r[1 - (r^2\omega^2/c^2)]^{1/2}. \tag{2}$$

This relationship was first given by Jennison (1964) who showed that it followed from a logical argument using transponders and telemetered clocks. Recent experiments by Davies and Jennison have confirmed that transponders in rotation act as predicted, they compute that a disc 10 cm in diameter rotating at 150 rad s^{-1} exhibits a relative contraction of 5 parts in 10^{17} in agreement with equation (2).

We note that if ω is maintained constant and r' is measured by radar it is evident from equation (2) that as r increases from zero, r' increases at first but reaches a maximum when $r = c/\omega\sqrt{2}$, ie where $r' = \frac{1}{2}c/\omega$, and then approaches zero again in the limit $r\omega \to c$.

The angular velocity, ω', measured by the rotating observer may be obtained by utilizing the common assumption of relativity theory that observers in two domains assign to each other equal but opposite velocities. Hence, the velocity of the rotating observer, v, as measured by an observer in S is given by

$$|v| = r\omega = r'\omega'.$$

Therefore, by equation (2)

$$\omega' = \omega[1 - (r^2\omega^2/c^2)]^{-1/2}.$$

This relationship may also be obtained from an experimental result by Davies and Jennison to an accuracy of about one part in 10^{16}. It differs from the theoretical result of Irvine (1964) who obtained $\Omega = \omega[1 - (r^2\omega^2/c^2)]^{-1}$ and claimed that this would be 'the angular velocity measured by local experiments by an observer in the rotating system'.

Irvine's result is clearly inconsistent with the results of the experiments by Davies and Jennison. The angular velocity that he derives does not appear to be an experimentally observable quantity.

3. Piecemeal measurements of ray paths in synchronously rotating systems

Figure 1 is drawn in an inertial frame, S, at rest relative to the centre, O, of a synchronously rotating disc and all distances on the diagram are measured in this frame. AX depicts the straight line path of a light ray in the inertial system, S, and we shall

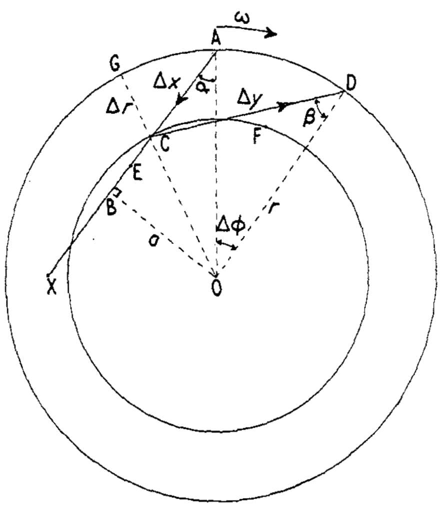

Figure 1. Ray paths (AC, CD), between concentric circles inscribed upon a synchronously rotating disc, according to observers in the inertial frame of the centre of the disc.

assume that an observer rotating with the disc wishes to determine the length of the corresponding curved path of the light ray in the frame rotating with the disc. The length of the path in the rotating frame may be determined by the rotating observer if he uses a method of successive radar measurements. The rotating observer crosses the path of the incident light ray once every revolution of the disc. As he crosses its path he makes a short-range radar measurement along the direction of propagation of the light ray. By making an infinite number of such infinitesimal, contiguous measurements along the light path the rotating observer may establish the length of the light path in the rotating system. This is the method used in the mathematical treatment given below and is precisely in agreement with the treatment of Arzeliès (1966, p 219). An alternative approach leading to an identical result would be to fire a very short laser pulse across the surface of the rotating disc such that the path of the pulse would be indelibly marked on its surface. The length of this light path could then be determined by the rotating observer if he again made an infinite number of infinitesimal, contiguous radar measurements along the path and then summed them.

From figure 1, X is any point on AB, or AB produced, B being the point of closest approach of the light ray to the centre. The rotating observer, when coincident with A, sends out a light signal along AC and receives the reflected signal from C when he has moved to point D in the inertial system of the centre but has remained at rest in his rotating system. He then goes to point C and sends a signal to a point E and receives it back when he has moved to some other point F in the inertial system of the centre. He repeats the process many times until he has traversed AX in a series of steps. The time taken for the signal to travel ACD as measured by the rotating observer's own clock, multiplied by the velocity of light, c, may be interpreted by the rotating observer as twice the radar distance to C. All other measurements are similarly interpreted and the distance from A to X in the rotating frame may be calculated by summing all the contiguous distance measurement. This procedure is equivalent to the standard experimental practice of calibrating extended distances by contiguous *étalons* (Ditchburn 1952).

It can readily be shown from the geometry of figure 1 that if $\Delta x \equiv AC$, $\Delta y \equiv CD$, $\Delta r \equiv GC$ and $\Delta t \equiv \Delta\phi/\omega$ are considered as infinitesimals then

$$\Delta x = r(r^2 - a^2)^{-1/2}\,\Delta r \tag{3}$$

$$r\Delta\phi = |(\Delta y^2 - \Delta r^2)^{1/2} \mp (\Delta x^2 - \Delta r^2)^{1/2}| \tag{4}$$

and

$$\Delta y = c\Delta t - \Delta x. \tag{5}$$

Substituting for Δx and Δy from equations (3) and (5) into equation (4), we find that

$$\tfrac{1}{2}c\Delta t = r[1 \pm (a\omega/c)](r^2 - a^2)^{-1/2}[1 - (r^2\omega^2/c^2)]^{-1}\Delta r. \tag{6}$$

But, we know that $\Delta\tau = \Delta t(1 - r^2\omega^2/c^2)^{1/2}$ from equation (1), therefore,

$$\tfrac{1}{2}c\Delta\tau = [1 \pm (a\omega/c)]r(r^2 - a^2)^{-1/2}[1 - (r^2\omega^2/c^2)]^{-1/2}\Delta r. \tag{7}$$

Defining $\frac{1}{2}c\Delta\tau$ as an incremental distance $\Delta L'$ in the rotating frame (ie as the radar distance to C from the mid-point of the arc AD according to the rotating observer) and letting $\Delta L' \to 0$ as $\Delta r \to 0$ we have that $L'' \equiv AB$ in the rotating system is given by

$$L'' = \sum_i \lim_{\Delta r_i \to 0} \Delta L_i'$$

hence

$$L'' = \int_a^r [1 \pm (a\omega/c)]r(r^2 - a^2)^{-1/2}[1 - (r^2\omega^2/c^2)]^{-1/2}\,dr$$

which, upon integration, yields

$$L'' = (c/\omega)[1 \pm (a\omega/c)]\sin^{-1}\{(\omega/c)(r^2 - a^2)^{1/2}[1 - (\omega^2a^2/c^2)]^{-1/2}\}.$$

If we construct BO produced and let it meet the circle of radius c/ω at a point H, then $BH = (c/\omega) \pm a$. Also, m is defined to be the length, in the inertial system of the centre, of the chord of the circle on BH as diameter, joining B and a point of intersection A of that circle with the circle centre O and radius r (figure 2). Hence, by geometry, we may express

$$m = [1 \pm (\omega a/c)](r^2 - a^2)^{1/2}[1 - (\omega^2a^2/c^2)]^{-1/2}$$

giving

$$L'' = [(c/\omega) \pm a]\sin^{-1}\{m/[(c/\omega) \pm a]\} \tag{8}$$

which is an arc of a circle of radius $\frac{1}{2}[(c/\omega) \pm a]$ with centre $\frac{1}{2}[(c/\omega) \mp a]$ from O.

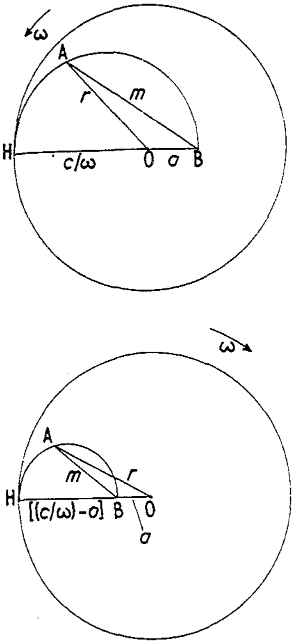

Figure 2. The geometry of a ray path according to piecemeal measurements made by observers upon the rotating disc. The ray path follows the circular arc HAB in the upper diagram and BAH in the lower diagram.

The sign depends upon the direction of rotation. We have so far measured $L'' \equiv \text{AB}$ in the rotating system but BX may similarly be measured, thus enbabling AX to be obtained. It is important to note that L'' is not the minimum distance between A and B; what we have measured is the path distance along a ray of light, ie along a null geodesic, by contiguous short-range measurements.

It is also instructive to calculate the angle of emission and angle of reception of the light signal used for each of the piecemeal measurements. In S, from figure 1, these angles are given by $\angle\text{OAC} = \alpha$ and $\angle\text{ODC} = \beta$ respectively. Therefore,

$$\sin\alpha = a/r \tag{9}$$

and

$$\cos\beta = \Delta r/\Delta y. \tag{10}$$

Substituting for $c\Delta t$ from equation (5) into equation (6) gives

$$\Delta y = 2r[1 \pm a(\omega/c)](r^2 - a^2)^{-1/2}[1 - (r^2\omega^2/c^2)]^{-1}\Delta r - \Delta x.$$

Substituting for Δx from equation (3) and using equation (10) gives

$$\sin\beta = r^{-1}\{2r^2(\omega/c) \pm a[1 + (r^2\omega^2/c^2)]\}[1 + (r^2\omega^2/c^2) \pm (2a\omega/c)]^{-1}. \tag{11}$$

Denoting the angle between the velocity vector at A and the direction of motion of the emitted light signal from A by θ_{A} and denoting the angle between the velocity vector at

D and the direction of motion of the received light signal at D by θ_D we have from figure 1 and equations (9) and (11) that

$$\cos\theta_A = \mp a/r \tag{12}$$

and

$$\cos\theta_D = r^{-1}\{(2r^2\omega/c) \pm a[1+(r^2\omega^2/c^2)]\}[1+(r^2\omega^2/c^2)\pm(2a\omega/c)]^{-1}. \tag{13}$$

The rotating observer will measure the angles of emission and reception as θ'_A and θ'_D which are related to θ_A and θ_D through the aberration formula (see, for example, Ditchburn 1952, Jennison 1963, 1964):

$$\cos\theta' = [\cos\theta - (v/c)][1-(v/c)\cos\theta]^{-1}. \tag{14}$$

Therefore, by equations (12)–(14) and remembering that $v = r\omega$, we find that

$$\cos\theta'_A = -r^{-1}(\omega r^2 \pm ac)(c \pm a\omega)^{-1} \tag{15}$$

and

$$\cos\theta'_D = r^{-1}(\omega r^2 \pm ac)(c \pm a\omega)^{-1} \tag{16}$$

hence,

$$\theta'_D = \theta'_A + \pi. \tag{17}$$

According to the rotating observer therefore, the light signals used to measure infinitesimal piecemeal distances on the disc are emitted and received in one and the same direction.

4. Piecemeal measurements of geodesics in synchronously rotating systems

Here we adopt the simple definition of a geodesic as the shortest distance between two points measured according to the principle stated in the introduction. Firstly, let us measure the minimum distance $\Delta r'$ between two concentric circles of inertial radii r and $r+\Delta r$, centre O, according to a rotating observer situated $r+\Delta r$ from the centre of the rotating system. Again we shall perform our measurements by using a light signal emitted by the rotating observer at A, scattered at C and received again by the rotating observer when he has reached D in the inertial system of the centre (see figure 1). We seek the minimum distance between two concentric circles. Taking $\Delta x = \Delta y$ in S makes the proper time interval between emission and reception of the light signals, according to the rotating observer, also a minimum. Hence, $\Delta\tau$ must be a minimum. Differentiating equation (7) with respect to a and setting $d\Delta\tau/da = 0$ we find that (taking the lower sign in all cases)

$$a = \omega r^2/c \tag{18}$$

for minimum path distance. Substituting for a from equation (18) into equation (7) we find that

$$\tfrac{1}{2}c\Delta\tau = \Delta r$$

but, by definition, $\frac{1}{2}c\Delta\tau \equiv \Delta r'$ where $\Delta r'$ is the infinitesimal separation of the two radii as measured by the rotating observer. Hence,

$$\Delta r' = \Delta r. \tag{19}$$

Also, substituting $a = \omega r^2/c$ into equations (15) and (16) gives $\cos\theta'_A = \cos\theta'_D = 0$ and hence, $\theta'_A = -\frac{1}{2}\pi$, $\theta'_D = \frac{1}{2}\pi$ by equation (17). The rotating observer therefore measures the minimum distance between concentric circles to be perpendicular to the velocity vector and to be equal in magnitude to the inertial minimum distance between concentric circles. Let us suppose that the rotating observer repeats these measurements along the length of a single inertial radius. This piecemeal 'geodesic' radius G''_r, measured by the rotating observer will be given by

$$G''_r = \sum_i \lim_{\Delta r_i \to 0} \Delta r'_i$$

which, on account of equation (19), can be written in the form

$$G''_r = \sum_i \lim_{\Delta r_i \to 0} \Delta r_i = r,$$

the inertial radius. Radial 'geodesics' are therefore always at right angles to the velocity vector and equal in length to the radius measured by an observer at rest relative to the inertial frame of reference, S, of the centre. G''_r is the *minimum* distance between a point on the rotating disc and the centre of the disc according to piecemeal measurements made on the disc. It differs from the piecemeal length of a light path between the same point and the centre which may be obtained by setting $a = 0$ in equation (8), giving $L''_r = (c/\omega)\sin^{-1}(r\omega/c)$. L''_r is the length measured piecemeal along the path of a null geodesic, whereas G''_r is the length measured piecemeal along the path of a geodesic.

Geodesics between any two points on a rotating disc may be obtained by referring to figure 2. To an observer at A the piecemeal geodesic $\Delta G'$ between A and B will be along AB. But, the element of the geodesic at A between A and O is, as we have just proved, equal to Δr therefore,

$$\cos(\angle OAB) = \Delta r'/\Delta G' = \Delta r/\Delta G' \tag{20}$$

and, from figure 2,

$$a^2 = r^2 + m^2 - 2rm\cos(\angle OAB). \tag{21}$$

But, by geometry

$$m = [1 + (\omega a/c)](r^2 - a^2)^{1/2}[1 - (\omega^2 a^2/c^2)]^{-1/2}$$

which, when substituted into equation (21), gives

$$\cos(\angle OAB) = r^{-1}(r^2 - a^2)^{1/2}[1 - (\omega^2 a^2/c^2)]^{-1/2}$$

or, by equation (20),

$$\Delta G' = r(r^2 - a^2)^{-1/2}[1 - (\omega^2 a^2/c^2)]^{1/2}\Delta r. \tag{22}$$

The geodesic, G'', between A and B in the rotating system is given by

$$G'' = \sum_i \lim_{\Delta r_i \to 0} \Delta G'_i$$

hence

$$G''=\int_a^r r(r^2-a^2)^{-1/2}[1-(\omega^2a^2/c^2)]^{1/2}\,dr$$

which, upon integration, yields

$$G''=[1-(\omega^2a^2/c^2)]^{1/2}(r^2-a^2)^{1/2} \tag{23}$$

which reduces to $G''_r=r$ for the length of the piecemeal geodesic radius when a is set equal to zero.

5. Conclusions

The preceding sections clearly indicate the necessity for precise definitions of how and what one is measuring in rotating systems. In § 2, which deals with 'measurements at a distance' by means of radar signals and involves no integration or summation of small piecemeal measurements, we have shown that the radius of a rotating disc, equation (2), as measured by an observer on the disc, quite clearly differs from that measured in the frame of reference of an observer stationary with respect to the centre of the disc, in agreement with measurements by Davies and Jennison. But, on the other hand, if radar techniques are employed to measure the length of a geodesic radius on the disc by means of summing small piecemeal sections of the geodesic we find in § 4 that this results in an identical length to that measured by an observer in a frame stationary with respect to the centre. What, therefore, do we mean by 'radius' in the context of a rotating system? From what has already been stated it would seem logical to expect that all electromagnetic effects observed or measured by a single observer fixed at some point in the rotating system must be interpreted as in § 2, ie in terms of the proper time of the observer (Jennison has extended this to mechanical effects). Therefore, all distance measurements made by the rotating observer to points outside the rotating system, provided that these distances, d, when measured in the inertial frame of the centre are large compared to the radius, r, of the rotating system, will be equal to $d[1-(r^2\omega^2/c^2)]^{1/2}$. Thus, the 'universe' according to the rotating observer will be reduced in radius by the factor $[1-(r^2\omega^2/c^2)]^{1/2}$ and will, in the limit $r\omega\to c$, be reduced to a singularity. This result may be applicable to rotating field models of the fundamental particles and may then be associated with the spatial uncertainty of some interactions.

It can also be seen from § 4, equation (23), that the 'geodesics' of the disc to rotating observers define a non-Euclidean space. The relationship between the geometry of the disc according to observers upon it and those in the inertial frame of the centre is only of use when it is necessary to perform transformations from the rotating system to the inertial frame of reference of the centre, or vice versa. These transformations are never required when proper measurements are made by single observers.

The analysis of § 3 points out the difference between geodesics and null geodesics in the rotating system and indicates quite clearly the difference in length between them when they are measured by summing an infinite number of infinitesimal piecemeal measurements. It is concluded in § 3 that the light signals used to measure infinitesimal piecemeal distances on the disc are emitted and received, according to the rotating observer, in one and the same direction, just as the light signals used for measurements in ordinary flat Euclidean space would be. Light signals used to measure large instances, on the other hand, are emitted and received in very different directions and it is here that one of the major differences arises between measurements performed by observers in constrained circular motion and measurements performed by observers in the unconstrained inertial motion to which the Lorentz transformations may be applied for extended periods of time. This analysis shows, however, that these same transformations may be applied instantaneously to derive the correct experimental results for systems in rotation.

Note added in proof. We note that in Davies and Jennison (1975) on page 1394, line 10 'equation (1)' should, of course, read 'the first equation'.

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